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# KKM Property and Fixed Point Theorems*

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## 1. INTRODUCTION

Recently, the famous Knaster–Kuratowski–Mazurkiewicz theorem [9] and Fan's lemma [5] have widely been used as very versatile tools in modern nonlinear analysis (see, for example, [1–3, 8, 10, 18, 19]).

It is well known that if $T$ is a KKM mapping from a convex subset $X$ of a topological vector space $E$ into $2^E$ with closed values then the family $\{Tx\colon x \in X\}$ has the finite intersection property. Many authors had generalized the above result to the following form.

Let $X$ be a convex space, $Y$ a topological space, and let $S, T$ be two set-valued mappings from $X$ into $2^Y$ such that for any finite subset $\{x_1, x_2, \dots, x_n\}$ of $X, T(\mathrm{co}\{x_1, \dots, x_n\}) \subset \bigcup_{i=1}^{n} Sx_i$, then the family $\{\overline{Sx}\colon x \in X\}$ has the finite intersection property under some assumptions of $T$ (for example, one may see [11–14]).

In this paper, we shall use the above results to define a family of set-valued mappings from a set $X$ into a topological space $Y$. We get some fixed point theorems and coincidence theorems which extend the results of

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[7, 14, 15]. We also get a generalized Fan's matching theorem and a generalized KKM theorem. As applications, we use these results to study the existence problem of solutions for some generalized variational inequalities.

## 2. PRELIMINARIES

Let $X$ and $Y$ be two sets, and let $T\colon X \to 2^Y$ be a set-valued mapping where $2^Y = \{B \subset Y\colon B \text{ is nonempty}\}$, we shall use the following notions in the sequel:

(i) $TA = \bigcup_{x \in A} Tx$ for $A \subset X$;

(ii) $T^{-1}y = \{x \in X : y \in Tx\}$ for $y \in Y$;

(iii) $T^{-1}B = \bigcup_{y \in B} T^{-1}y$ for $B \subset Y$.

For the case that $X$ and $Y$ are two topological spaces,

(iv) $T$ is said to be upper semi-continuous (u.s.c.) if for each closed subset $B$ of $Y, T^{-1}B$ is closed in $X$;

(v) $T$ is said to be closed if the graph $\Gamma_T = \{(x, y) \in X \times Y : y \in Tx\}$ is a closed subset of $X \times Y$; and

(vi) $T$ is said to be compact if the image $TX$ of $X$ under $T$ is contained in a compact subset of $Y$.

Assume that $X$ is a convex subset of a linear space and $Y$ is a topological space. If $S, T\colon X \to 2^Y$ are two set-valued mappings such that $T(\operatorname{co} A) \subset SA$ for each finite subset $A$ of $X$, then we call $S$ a generalized KKM mapping w.r.t. $T$, where $\operatorname{co} A$ denotes the convex hull of $A$. Let $T\colon X \to 2^Y$ be a set-valued mapping such that if $S\colon X \to 2^Y$ is a generalized KKM mapping w.r.t. $T$ then the family $\{\overline{Sx}\colon x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of $Sx$), then we say that $T$ has the KKM property. Denote $\mathrm{KKM}(X, Y) = \{T\colon X \to 2^Y | T$ has the KKM property$\}$.

*Remark.* (1) Generalized KKM mappings were first introduced by Park [13], and followed by some others.

(2) A closed-valued generalized KKM set-valued mapping w.r.t. some set-valued mapping may not have the finite intersection property (f.i.p.).

For example, if $X = [0,1]$ with the usual topology and $S, T\colon X \to 2^X$ are defined as

$$Sx = Tx = \begin{cases} \{0\}, & x \in \left[0, \frac{1}{2}\right); \\ \{0,1\}, & x = \frac{1}{2}; \\ \{1\}, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

then

(i) $Sx$ is closed (compact) for each $x \in X$;

(ii) $T$ is compact, closed (hence u.s.c.); and

(iii) $T(\mathrm{co}\, A) \subset SA$ for each finite subset $A$ of $X$,

however, the family $\{Sx\colon x \in X\}$ has no f.i.p.

A convex space $S$ is a convex set (in a linear space) with any topology that induces the Euclidean topology on the convex hull of its finite subset. For each finite subset $A$ of $X$, the convex hull of $A$ is called a polytope of $X$. Let $Y$ be a topological space, a subset $B$ of $Y$ is compactly closed if for any compact subset $K$ of $Y$, $B \cap K$ is closed in $K$.

Let $\mathbf{F}$ be a class of set-valued mappings. Denote

$\mathbf{F}(X,Y) = \{T\colon X \to 2^Y | T \in \mathbf{F}\}$

$\mathbf{F}_c = \{T_n T_{n-1} \ldots T_1 | T_i \in \mathbf{F},\ i = 1,2,\ldots,n$ for some $n\}$, that is, the set of finite composites of mappings in $\mathbf{F}$.

If $X$ and $Y$ are two topological spaces, we note

$\mathbf{C}(X,Y) = \{s\colon X \to Y | s$ is a single-valued continuous function$\}$;

$\mathbf{K}(X,Y) = \{T\colon X \to 2^Y | T$ is u.s.c. with compact convex values and $Y$ is convex$\}$;

$\mathbf{V}(X,Y) = \{T\colon X \to 2^Y | T$ is u.s.c. with compact acyclic values$\}$ (see [15]); and

$\mathbf{A}(X,Y) = \{T\colon X \to 2^Y | T$ is u.s.c. and approachable with compact values$\}$ (see [3]).

The following abstract class $\mathfrak{A}$ of set-valued mappings is introduced by Park [14].

A class $\mathfrak{A}$ of set-valued mappings is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbf{C}$ of single-valued continuous functions;

(ii) each $T \in \mathfrak{U}_c$ is u.s.c. with compact values; and

(iii) for any polytope $\Delta$, each $T \in \mathfrak{A}_c(\Delta, \Delta)$ has a fixed point.

Denote $\mathfrak{A}_c^{\kappa}(X,Y) = \{T\colon X \to 2^Y |$ for any compact subset $K$ of $X$, there is a $F \in \mathfrak{A}_c(K,Y)$ such that $Fx \subset Tx$ for each $x \in K\}$.

## 3. MAIN RESULTS

In the sequel, all topological spaces are supposed to satisfy the Hausdorff separation property.

We first prove the following lemma:

LEMMA 1. *Let $X$ be a convex subset of a topological vector space $E$, $V$ a symmetric convex open neighborhood of $0$ in $E$. If $T \in \mathrm{KKM}(X, X)$ is compact, then there is $x_v \in X$ such that $(x_v + V) \cap Tx_v \neq \emptyset$.*

*Proof.* Suppose that $(x + V) \cap Tx = \emptyset$ for each $x \in X$. Let $X_1 = \overline{TX}$ then $X_1$ is a compact subset of $X$ since $T$ is compact. Define the mapping $S: X \to 2^X$ by

$$Sx = X_1 \setminus \left(x + \tfrac{1}{4}V\right) \qquad \text{for each } x \in X.$$

We have

(a) $Sx$ is closed in $X$ for each $x \in X$; and

(b) $S$ is a generalized KKM mapping w.r.t. $T$.

To prove (b), it suffices to show that for any finite subset $A = \{x_1, x_2, \dots, x_n\}$ of $X, T(\mathrm{co}\ A) \subset SA$.

It is trivial for the case that $n = 1$. We now consider $n \geq 2$.

*Case* (i). If there is some $i$, $2 \leq i \leq n$, such that $(x_i + \frac{1}{4}V) \cap (x_1 + \frac{1}{4}V) = \emptyset$, then $T(\mathrm{co}\ A) \subset X_1 = Sx_i \cup Sx_1 \subset SA$.

*Case* (ii). Suppose that $(x_i + \frac{1}{4}V) \cap (x_1 + \frac{1}{4}V) \neq \emptyset$ for each $2 \leq i \leq n$. We have $x_1 \in x_i + \frac{1}{2}V$, for each $2 \leq i \leq n$. Hence $x_1 \in x + \frac{1}{2}V$ for each $x \in \mathrm{co}\ A$. This implies $x_1 + \frac{1}{2}V \subset \bigcap_{x \in \mathrm{co}\ A}(x + V)$ and so $T(\mathrm{co}\ A) \subset Sx_1 \subset SA$.

Since $T \in \mathrm{KKM}(X, X)$, hence the family $\{Sx: x \in X\}$ has f.i.p. and hence $\bigcap_{x \in X} Sx \neq \emptyset$. Let $x_0 \in \bigcap_{x \in X} Sx$, then $x_0 \in Sx_0$. This contradicts the definition of $S$.

Using Lemma 1, we get our main theorem:

THEOREM 2. *Let $X$ be a convex subset of a locally convex space $E$, and $T \in \mathrm{KKM}(X, X)$. If $T$ is compact and closed, then $T$ has a fixed point in $X$.*

*Proof.* Suppose $\{V_\alpha: \alpha \in \Lambda\}$ is a local basis of $E$ such that $V_\alpha$ is symmetric, open, and convex for each $\alpha \in \Lambda$.

By Lemma 1, for each $\alpha \in \Lambda$, there exists $x_\alpha \in X$ such that $Tx_\alpha \cap (x_\alpha + V_\alpha) \neq \emptyset$. That is, there exists $y_\alpha \in Tx_\alpha$ such that $y_\alpha \in x_\alpha + V_\alpha$ for each $\alpha \in \Lambda$. Since $T$ is compact, without loss of generality, we may assume that $y_\alpha$ converges to some $\hat{y}$ in $Y$, then $x_\alpha$ also converges to $\hat{y}$. Since $T$ is closed, we have $\hat{y} \in T\hat{y}$.

We now list some properties of the KKM family.

PROPOSITION 3. *Let $X$ be a convex subset of a linear space, and let $Y, Z$ be two topological spaces. Then*

(i) *$T \in \mathrm{KKM}(X, Y)$ if and only if $T|_{\Delta} \in \mathrm{KKM}(\Delta, Y)$ for each polytope $\Delta$ in $X$;*

(ii) *if $T \in \mathrm{KKM}(X, Y)$ and $f \in \mathbf{C}(Y, Z)$, then $fT \in \mathrm{KKM}(X, Z)$;*

(iii) *if $Y$ is a normal space, $X$ a convex space, $\Delta$ a polytope in $X$, and if $T\colon \Delta \to 2^Y$ is a set-valued mapping such that for each $f \in \mathbf{C}(Y, \Delta)$, $fT$ has a fixed point in $\Delta$, then $T \in \mathrm{KKM}(\Delta, Y)$; and*

(iv) *if $X$ is a convex space, then $\mathfrak{A}_c^{\kappa}(X, Y) \subset \mathrm{KKM}(X, Y)$.*

*Proof.* Part (i) is obvious. To prove (ii), let $S\colon X \to 2^Z$ be a generalized KKM mapping w.r.t. $fT$ such that $Sx$ is closed for each $x \in X$. Then, for any finite subset $\{x_1, x_2, \dots, x_n\}$ of $X$, since $S$ is a generalized KKM mapping w.r.t. $fT$, we have

$$fT\left(\mathrm{co}\{x_1, x_2, \dots, x_n\}\right) \subset \bigcup_{i=1}^{n} Sx_i,$$

and hence $T(\mathrm{co}\{x_1, x_2, \dots, x_n\}) \subset \bigcup_{i=1}^{n} f^{-1}Sx_i$. Therefore, $f^{-1}S$ is a generalized KKM mapping w.r.t. $T$. Since $T \in \mathrm{KKM}(X, Y)$, hence the family $\{f^{-1}Sx\colon x \in X\}$ has f.i.p., and hence the family $\{Sx\colon x \in X\}$ has f.i.p. This shows that $fT \in \mathrm{KKM}(X, Z)$ and we complete the proof of (ii).

The proof of (iii) is similar to the proof of [11, Lemma 2] and we omit it.

Part (iv) is a direct consequence of [14, Corollary 2].

*Remark.* (1) Since $\mathbf{C}$, $\mathbf{K}$, and $\mathbf{V}$ are examples of $\mathfrak{A}$, hence, by Proposition 3(iv), $\mathbf{K}_c \subset \mathrm{KKM}$ and $\mathbf{V}_c \subset \mathrm{KKM}$.

(2) The following example shows that $\mathrm{KKM} \supsetneq \mathfrak{U}_c^{\kappa}$, hence Theorem 2 properly extends Theorems 3(iii) of [14].

EXAMPLE. Let $X = Y = [0, 1]$ with the usual topology, and let $T\colon X \to 2^Y$ be defined as

$$Tx = \begin{cases} \left\{\left|\sin\dfrac{1}{x}\right|\right\}, & x \in (0, 1]; \\ \{0\}, & x = 0. \end{cases}$$

Then $T \in \mathrm{KKM}(X, Y)$, however $T \notin \mathfrak{A}_c^{\kappa}(X, Y)$.

By Proposition 3 and Theorem 2, we get the following coincidence theorem which extends a result of Park, Singh and Watson [15, Theorem 1] and the main result of Granas and Liu [7].

THEOREM 4. *Let $X$ be a convex space, and let $T, G\colon X \to 2^Y$ be two set-valued mappings satisfying*:

(i) $T \in \mathrm{KKM}(X, Y)$ *is compact and closed*;
(ii) *for each* $y \in GX$, $G^{-1}y$ *is convex*; *and*
(iii) $\{\operatorname{int} Gx\colon x \in X\}$ *covers* $\overline{TX}$.

*Then there exists* $x_0 \in X$ *such that* $Tx_0 \cap Gx_0 \neq \varnothing$.

*Proof.* Since $T$ is compact and $\overline{TX} \subset \bigcup_{x \in X} \operatorname{int} Gx$, hence there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of $X$ such that $\overline{TX} \subset \bigcup_{i=1}^{n} \operatorname{int} Gx_i$. Let $\Delta = \operatorname{co}\{x_1, x_2, \dots, x_n\}$, and let $\{\alpha_i\}_{i=1}^n$ be the partition of the unity subordinated to $\{\operatorname{int} Gx_i\colon i = 1, 2, \dots, n\}$. Define $f\colon \overline{TX} \to \Delta$ by

$$f(y) = \sum_{i=1}^{n} \alpha_i(y) x_i \qquad \text{for each } y \in \overline{TX}$$

then $f \in \mathbf{C}(\overline{TX}, \Delta)$. By Proposition 3(i), $T|_\Delta \in \mathrm{KKM}(\Delta, \overline{TX})$, and by Proposition 3(ii) and Theorem 2, $fT$ has a fixed point in $\Delta$, that is, there exists $x_0 \in \Delta$ such that $x_0 \in fTx_0$. Let $y_0 \in Tx_0$ such that $x_0 = f(y_0)$, and let $I(y_0) = \{i \in \{1, 2, \dots, n\}\colon \ \alpha_i(y_0) > 0\}$. Then $x_0 = \sum_{i \in I(y_0)} \alpha_i(y_0) x_i$. If $\alpha_i(y_0) > 0$, then $y_0 \in \operatorname{int} Gx_i \subset Gx_i$, and then $x_i \in G^{-1}y_0$. So $x_0 \in \operatorname{co}\{x_i\colon i \in I(y_0)\} \in \operatorname{co}(G^{-1}y_0) = G^{-1}y_0$, since $G^{-1}y_0$ is convex. This implies $x_0 \in G^{-1}y_0$ and so $y_0 \in Gx_0$. Thus $y_0 \in Tx_0 \cap Gx_0$ and we complete our proof.

We now use a result of Ben-El-Mechaiekh and Deguire to get some fixed point theorems and a generalized Fan's matching theorem.

PROPOSITION 5. *Let $X$ be a convex subset of a locally convex space $E$, $Y$ a compact subset of a topological vector space $F$, and let* $T \in \mathrm{KKM}(X, Y)$ *be closed. Then for any* $G \in \mathbf{A}_c(Y, X)$, *$TG$ has a fixed point in $Y$* (*and hence $GT$ has a fixed point in $X$*).

*Proof.* Since $T \in \mathrm{KKM}(X, Y)$, hence for any $f \in \mathbf{C}(Y, X)$, by Proposition 3(ii), $fT \in \mathrm{KKM}(X, X)$, and hence, by Theorem 2, $fT$ has fixed point in $X$. That is, $\Gamma_f \cap \Gamma_{T^{-1}} \neq \varnothing$. So, by [3, Corollary 7.5], we have $\Gamma_G \cap \Gamma_{T^{-1}} \neq \varnothing$. That is, $TG$ has a fixed point in $Y$.

COROLLARY 6. *Let $X$ and $Y$ be defined as in Proposition* 5, *and let $Z$ be a subset of a topological vector space $L$. If* $T \in \mathrm{KKM}(X, Y)$ *is closed, then for each* $G \in \mathbf{A}_c(Y, Z)$, $GT \in \mathrm{KKM}(X, Z)$.

*Proof.* By Proposition 3(i), it suffices to show that $GT|_\Delta \in \mathrm{KKM}(\Delta, Z)$, for each polytope $\Delta$ in $X$. Since for any $f \in \mathbf{C}(Z, \Delta)$, $fG \in \mathbf{A}_c(Y, \Delta)$, so, by Proposition 5, $fGT|_\Delta$ has a fixed point in $\Delta$. Thus, by Proposition 3(iii), $GT|_\Delta \in \mathrm{KKM}(\Delta, Z)$.

*Remark.* It also can be shown, under some suitable modification, that for any $T \in \mathrm{KKM}(X, Z)$ and $G \in \mathbf{A}_c(Y, X)$, we have $TG \in \mathrm{KKM}(Y, Z)$.

Using the above result, we have the following theorem, which is a generalized Fan's matching theorem:

THEOREM 7. *Let $X$ be a compact convex subset of a locally convex space, and let $\{A_\nu: \nu \in I\}$ be a finite family of closed subsets of $X$ indexed by a set $I$, such that $\bigcup_{\nu \in I} A_\nu = X$. If $T \in \mathrm{KKM}(X, X)$ is compact and closed, then for any family $\{x_\nu\}_{\nu \in I}$ of points of $X$, indexed by the same set $I$, there exists a nonempty subset $J$ of $I$ such that*

$$T\big(\mathrm{co}\{x_\nu: \nu \in J\}\big) \cap \left(\bigcap_{\nu \in J} A_\nu\right) \neq \varnothing.$$

*Proof.* For each $x \in X$, let $I(x) = \{\nu \in I: x \in A_\nu\}$, then $I(x) \neq \varnothing$ for each $x \in X$ since $\{A_\nu\}_{\nu \in I}$ covers $X$.

Define $F: X \to 2^X$ by

$$Fx = \mathrm{co}\{x_\nu: \nu \in I(x)\} \qquad \text{for each } x \in X.$$

It is clear that $Fx$ is a nonempty compact convex subset of $X$. For each $x \in X$, let $U(x) = X \setminus \cup\{A_\nu: \nu \notin I(x)\}$, then $U(x)$ is an open neighborhood of $x$. If $z \in U(x)$, then $Fz \subset Fx$. This shows that $F$ is u.s.c., hence $F \in \mathbf{K}(X, X)$, and hence, by Proposition 4.1 of [3], $F \in \mathbf{A}(X, X)$. Now, by Proposition 5, there exists $x_0 \in X$ such that $x_0 \in TFx_0 = T(\mathrm{co}\{x_\nu: \nu \in I(x_0)\})$. So,

$$x_0 \in T\big(\mathrm{co}\{x_\nu: \nu \in I(x_0)\}\big) \cap \left(\bigcap_{\nu \in I(x_0)} A_\nu\right)$$

which completes the proof.

*Remark.* Fan [6, Theorem 1] proved Theorem 7 for the case that $T = i_x$ (the identity mapping of $X$).

## 4. APPLICATIONS

In this section, we shall use the results in Section 3 and the following generalized KKM theorem to get some generalized variational inequalities.

THEOREM 8. *Let $X$ be a convex space, $Y$ a topological space, let $S\colon X \to 2^Y$ be a set-valued mapping, and let $T \in \mathrm{KKM}(X, Y)$ such that*

(i) *for each compact subset $X'$ of $X$, $\overline{TX'}$ is compact in $Y$;*

(ii) *for each $x \in X$, $Sx$ is compactly closed in $Y$;*

(iii) *for any finite subset $A$ of $X$, $T(\mathrm{co}\, A) \subset SA$; and*

(iv) *there exist a nonempty compact convex subset $X_0$ of $X$ and a compact subset $K$ of $Y$ such that $\bigcap_{x \in X_0} Sx \subset K$.*

*Then $\bigcap_{x \in X} Sx \neq \emptyset$.*

*Proof.* Suppose that $\bigcap_{x \in X} Sx = \emptyset$, then $\bigcup_{x \in X} S^c x = Y$, and $\bigcup_{x \in X}(S^c x \cap K) = K$. Since $K$ is compact and $S^c x$ is compactly open, there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of $X$ such that $\bigcup_{i=1}^n (S^c x_i \cap K) = K$. By assumption (iv), $\bigcap_{x \in X_0} Sx \subset K$, we have $\bigcup_{x \in X_0} S^c x \supset K^c$. Now, let $X_1 = \mathrm{co}(X_0 \cup \{x_1, x_2, \dots, x_n\})$ then $X_1$ is a compact convex set and $\bigcup_{x \in X_1} S^c x = Y$. Hence $\bigcap_{x \in X_1} Sx = \emptyset$.

By (i), $\overline{TX_1}$ is compact in $Y$. Define $F\colon X_1 \to 2^Y$ by

$$Fx = Sx \cap \overline{TX_1} \qquad \text{for each } x \in X_1.$$

Then

(a) $Fx$ is closed (compact) in $Y$ for each $x \in X_1$, and

(b) $T(\mathrm{co}\, A) \subset FA$, for any finite subset $A$ of $X_1$.

Since $T|_{X_1} \in \mathrm{KKM}(X, Y)$, the family $\{Fx\colon x \in X_1\}$ has f.i.p., hence $\bigcap_{x \in X_1} Fx \neq \emptyset$, and hence $\bigcap_{x \in X_1} Sx \neq \emptyset$. This is a contradiction, and we complete our proof.

*Remark.* Condition (i) of Theorem 8 is weaker than the assumption that $T$ is compact.

As a consequence of the above theorem, we have the following theorem, which is generalized form of [16, Theorem 3].

THEOREM 9. *Let $X$ and $Y$ be defined as in Theorem 8, and let $T \in \mathrm{KKM}(X, Y)$ such that for each compact subset $X'$ of $X$, $\overline{TX'}$ is compact in $Y$. If $\psi, \varphi\colon X \times Y \to R$ are two real-valued functions satisfying:*

(i) $\psi(x, y) \leq 0$, *for each* $(x, y) \in \Gamma_T$;

(ii) *for fixed $x \in X$, the mapping $y \to \varphi(x, y)$ is lower semicontinuous* (l.s.c.) *on $A$ for each compact subset $A$ of $Y$;*

(iii) *for fixed $y \in Y$, the set $\{x \in X\colon \psi(x, y) > 0\}$ contains the convex hull of the set $\{x \in X\colon \varphi(x, y) > 0\}$; and*

(iv) *there exist a nonempty compact convex subset* $X_0$ *of* $X$, *and a compact subset* $K$ *of* $Y$ *such that for each* $y \in Y \setminus K$, *there is* $x \in X_0$ *such that* $\varphi(x, y) > 0$,

*then there exists* $\hat{y} \in K$ *such that* $\varphi(x, \hat{y}) \leq 0$ *for each* $x \in X$.

*Proof.* Define $F, S\colon X \to 2^Y$ by

$$Fx = \{y \in Y\colon \psi(x, y) \leq 0\}$$

and

$$Sx = \{y \in Y\colon \varphi(x, y) \leq 0\} \qquad \text{for each } x \in X.$$

By assumption (i), we have $\Gamma_T \subset \Gamma_F$, and by assumption (ii), $Sx$ is compactly closed for each $x \in X$. Condition (iii) implies that for each finite subset $A$ of $X$, $F(\text{co } A) \subset SA$, and then $T(\text{co } A) \subset SA$.

Condition (iv) is equivalent to Theorem 8(iv). So, all the conditions in Theorem 8 are satisfied, and so $\bigcap_{x \in X} Sx \neq \emptyset$. Take $\hat{y} \in \bigcap_{x \in X} Sx$, and we have $\varphi(x, \hat{y}) \leq 0$ for each $x \in X$.

For the case where $T$ is a single-valued mapping, we have the following corollary:

COROLLARY 10. *Let* $X$ *and* $Y$ *be defined as in Theorem* 8, *and let* $s \in \mathbf{C}(X, Y)$. *If* $\psi, \varphi\colon X \times Y \to R$ *are two real-valued functions such that*

(i) $\psi(x, sx) \leq 0$, *for each* $x \in X$;

(ii) *for fixed* $x \in X$, *the mapping* $y \to \varphi(x, y)$ *is* l.s.c. *on* $A$ *for each compact subset* $A$ *of* $Y$;

(iii) *for fixed* $y \in Y$, *the set* $\{x \in X\colon \psi(x, y) > 0\}$ *contains the convex hull of the set* $\{x \in X\colon \varphi(x, y) > 0\}$; *and*

(iv) *there exist a nonempty compact convex subset* $X_0$ *of* $X$, *and a compact subset* $K$ *of* $Y$ *such that for each* $Y \in Y \setminus K$ *there is* $x \in X$ *such that* $\varphi(x, y) > 0$,

*then there exist* $\hat{y} \in K$ *such that* $\varphi(x, \hat{y}) \leq 0$ *for each* $x \in X$.

*Remark.* For the case $X = Y$ and $s = i_X$ (the identity mapping of $X$), Corollary 10 reduces to [16, Theorem 3].

Let $X$ be a convex subset of a linear space, $Y$ a set, and let $\psi$, $\varphi\colon X \times Y \to R$ be two real-valued functions. For any $y \in Y$, $\psi$ is said to be $\varphi$-quasiconcave in $x$ if for any finite subset $A = \{x_1, x_2, \ldots, x_n\}$ of $X$, we have

$$\psi(x, y) \geq \min_{1 \leq i \leq n} \varphi(x_i, y) \qquad \text{for all } x \in \text{co } A.$$

It is clear that if $\varphi(x, y) \leq \psi(x, y)$ for each $(x, y) \in X \times Y$, and if for each $y \in Y$, the mapping $x \to \varphi(x, y)$ is quasiconcave (or the mapping $x \to \psi(x, y)$ is quasiconcave) then $\psi$ is $\varphi$-quasiconcave in $x$.

By the above definition, we have the following theorem concerning other inequalities.

THEOREM 11. *Let $X$ be a convex space, $Y$ a topological space, and let $T \in \mathrm{KKM}(X, Y)$ be compact. If $\psi, \varphi$: $X \times Y \to R$ are two real-valued functions satisfying that*

(i) *for each $x \in X$, the mapping $y \to \varphi(x, y)$ is* l.s.c. *on $Y$; and*

(ii) *$\psi$ is $\varphi$-quasiconcave in $x$ for each $y \in Y$,*

*then for each $\lambda \in R$ one of the following properties holds*:

(a) *there exists $\hat{y} \in Y$ such that*

$$\varphi(x, \hat{y}) \leq \lambda \qquad \text{for all } x \in X;$$

(b) *for some $(x_0, y_0) \in \Gamma_T$ we have*

$$\psi(x_0, y_0) > \lambda.$$

*Proof.* Let $\lambda \in R$. Since $T$ is compact, hence $\overline{TX}$ is compact in $Y$. Define $F, S$: $X \to 2^Y$ by

$$Fx = \{y \in \overline{TX}: \psi(x, y) \leq \lambda\}$$

and

$$Sx = \{y \in \overline{TX}: \varphi(x, y) \leq \lambda\} \qquad \text{for each } x \in X.$$

Suppose the conclusion (b) fails, then for each $(x, y) \in \Gamma_T$, $\psi(x, y) \leq \lambda$ and then $\Gamma_T \subset \Gamma_F$. Now, by (ii), we have $F(\mathrm{co}\, A) \subset SA$ for each finite subset $A$ of $X$. So, for any finite subset $A$ of $X$, $T(\mathrm{co}\, A) \subset SA$. Since $T \in \mathrm{KKM}(X, Y)$ and, by (i), $Sx$ is closed (compact), the family $\{Sx: x \in X\}$ has f.i.p. Hence $\bigcap_{x \in X} Sx \neq \emptyset$. Take $\hat{y} = \bigcap_{x \in X} Sx$, and we have

$$\varphi(x, \hat{y}) \leq \lambda \qquad \text{for each } x \in X.$$

For the case that $T$ is a single-valued function, we have the following corollary:

COROLLARY 12. *Let be a convex space, $Y$ a topological space, and let $s \in \mathbf{C}(X, Y)$ be compact. If $\psi, \varphi$: $X \times Y \to R$ are two real-valued functions* *satisfying conditions* (i) *and* (ii) *of Theorem* 11, *then for each* $\lambda \in R$, *one of the following properties holds*:

(a) *there exists* $\hat{y} \in Y$ *such that*

$$\varphi(x, \hat{y}) \leq \lambda \qquad \textit{for all } x \in X;$$

(b) *for some* $x_0 \in X$ *we have* $\psi(x_0, sx_0) > \lambda$.

*Remark*. (1) Theorems 5.1 and 5.2 of [7] are special cases of Theorem 11 and Corollary 12, respectively.

(2) The conclusion of Theorem 11 implies the inequality

$$\inf_{y \in Y} \sup_{x \in X} \varphi(x, y) \leq \sup_{(x, y) \in \Gamma_T} \psi(x, y),$$

and the conclusion of Corollary 12 implies the inequality

$$\inf_{y \in Y} \sup_{x \in X} \varphi(x, y) \leq \sup_{x \in X} \psi(x, sx),$$

which are the generalized forms of [18, Theorem 1].

We now present another result by using the KKM property directly.

THEOREM 13. *Let $X$ be a convex space, $Y$ a compact space. Let $\psi$, $\varphi$: $X \times Y \to R$ be two real-valued functions satisfying conditions* (i) *and* (ii) *of Theorem* 11, *and let* $\lambda \in R$. *If for any polytope* $\Delta$ *in* $X$ *and for* $s \in \mathbf{C}(Y, \Delta)$, *there exists* $y \in Y$ (*depends on* $s$) *such that* $\psi(sy, y) \leq \lambda$, *then there exists* $\hat{y} \in Y$ *such that*

$$\varphi(x, \hat{y}) \leq \lambda \qquad \textit{for all } x \in X.$$

*Proof.* Define $T, S$: $X \to 2^Y$ by

$$Tx = \{y \in Y: \psi(x, y) \leq \lambda\}$$

and

$$Sx = \{y \in Y: \varphi(x, y) \leq \lambda\} \qquad \text{for each } x \in X.$$

By assumption, for any polytope $\Delta$ in $X$ and for $s \in \mathbf{C}(Y, \Delta)$, $sT|_\Delta$ has a fixed point in $\Delta$. Hence, by Proposition 3(iii), $T \in$ KKM $(X, Y)$. The rest of the proof is similar to the proof of Theorem 11 and we omit it.

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